ON MOTIONS OF AN IDEAL FLUID WITH A PRESSURE DISCONTINUITY ALONG THE BOUNDARIES

(O DVIZHENII IDEAL'NOI ZHIDKOSTI S RAZRYVOM DAVLENIIA VDOL' GRANITSY)

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Pressure discontinuities occurring on the boundary of an ideal incompressible fluid occur, for example, in problems of the propagation of shock waves along the surface of a fluid. These problems are frequently solved in linearized form (the boundary perturbations are small because of the smallness of the ratio of the density of the air to the density of water) [1]. However, near the points where the discontinuity in pressure occurs (the front of the shock wave) the linearization is not valid, because the speed of the particles, as given by the linearized theory, increases indefinitely under these circumstances [1]. Below we shall consider the motion of an ideal fluid in the neighborhood of a pressure discontinuity, on its boundary without linearization of the problem. It is shown that the free boundary has a curved spiral form (it is assumed throughout that the fluid is ideal, incompressible, and not under the influence of gravity).

1. Consider the plane steady potential flow of a fluid in a domain bounded solely by a free surface *AFB*. Let x, y be the Cartesian coordinates in the plane of the motion, $z = x + iy$; with $z = \infty$ at the points *A* and *B* and $z = 0$ at the point *F*. Suppose that the velocity on the free surface is always directed from *A* to *B* and that its magnitude is v_1 on *AF* and v_2 on *FB* (at the point *F* there is a pressure discontinuity, and hence also a discontinuity in the speed), and the fluid occupies the domain on the right of the curve *AFB* (as seen by a moving observer traveling on *AFB* in the direction of the velocity vector on *AFB*).

Suppose that the complex potential $w(z) = \varphi + i\psi$ is zero at *F*; then

$$\varphi < 0 \quad \text{on } AF, \quad \varphi > 0 \quad \text{on } FB, \quad \psi = 0 \quad \text{on } AFB$$

and $\psi < 0$ in the demain of the motion. The function

$$Z(w) = \frac{\ln v_2 - \ln w'}{\ln v_2 - \ln v_1} \tag{1.1}$$

(which is introduced in following the method of Kirchhoff-Mitchell), the prime denoting the derivative with respect to z, is analytic in the half plane $\psi < 0$ and satisfies the boundary conditions

$$\operatorname{Re} Z = 1 \quad \text{for } \psi = 0,\ \varphi < 0; \qquad \operatorname{Re} Z = 0 \quad \text{for } \psi = 0,\ \varphi > 0$$

It is obvious that this function is precisely

$$Z(w) = -\frac{1}{\pi i} \ln \frac{w}{w_0} \tag{1.2}$$

where $w_0 > 0$ is an arbitrary constant, the branch of the logarithm being that for which $-\pi < \operatorname{Im} \ln w < 0$ for $\operatorname{Im} w < 0$. From (1.1) and (1.2), taking into account that $w(0) = 0$, one obtains the relation between w and z:

$$(1 - ia)\, v_2 z = w_0^{\,ia} w^{1-ia}, \qquad a = \frac{1}{\pi} \ln \frac{v_1}{v_2} \tag{1.3}$$

The complex potential is a power of z with a complex exponent, which degenerates to plane parallel flow when $a = 0$ (a change in w_0 corresponds to a rotation in the coordinate axes). From (1.3) it follows that the parts AF and FB of the boundary are logarithmic spirals of equations, respectively,

$$r_1 = be^{-[\theta + \pi(1+a^2)]/a}, \qquad r_2 = be^{-\theta/a} \tag{1.4}$$

$$b = \frac{w_0}{v_2\sqrt{1+a^2}} \exp\left(\frac{1}{a}\tan^{-1} a\right) \quad (z = re^{i\theta})$$

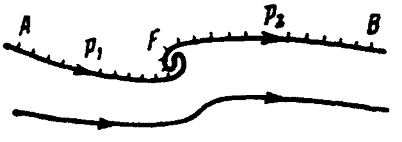

Fig. 1.

The case $v_1 > v_2$, $a > 0$ (and pressures $p_1 < p_2$) of the motion is illustrated in Fig. 1, where the nature of the streamlines is also shown. When $a < 0$ the spirals turn in the opposite sense, since in the neighborhood of F the flow is always toward the boundary with the smaller pressure. A particle moving on the boundary passes through F in a finite time, since the length of a bounded arc of a logarithmic spiral is always finite. The angle required to turn one spiral into the other in the domain of the motion, equals $\pi(1 + a^2) > \pi$; when $|a| = 1$ the fluid occupies the whole z-plane, cut along the spirals, and for $|a| > 1$ the domain of the motion consists of more than one sheet, while for $|a| \to +\infty$ the number of sheets increases indefinitely. Using Bernoulli's theorem and the Equation (1.3) for a, we obtain the following condition in order that the domain be simply covered (for $a < 1$, and $p_1 < p_2$, with ρ being the density of the fluid)

$$\frac{2(p_2 - p_1)}{\rho v_1^2} = 1 - \left(\frac{v_2}{v_1}\right)^2 \leqslant 1 - e^{-2\pi} \approx 0.998\,133 \tag{1.5}$$

The jump in pressure across a shock wave in an ideal gas with adiabatic exponent γ is given by the formula

$$p_2 - p_1 = \frac{2}{\gamma + 1} \rho_0 [(D - v_0)^2 - c_0^2] \tag{1.6}$$

where D, ρ_0, v_0, c_0 are, respectively, the wave propagation speed, the density, the particle speed, and the speed of sound in the gas before the shock. In the case of shock waves in air, propagating along the surface of water, when air and water are at rest in front of the wave (in a system of coordinates attached to the wave front one has $D = 0$, $v_0 = v_1$), the condition (1.5) is automatically satisfied. But in this case the effect under consideration is insignificant, since an increase in the distance of a thousand units from the point F on the boundary is achieved in less than 8.2 minutes, in view of (1.4). If the fluid moves in the direction of the wave with a certain speed (with respect to the coordinate system attached to the shock, $v_0 > v_1$), then the spiraling effect is stronger, and it increases considerably with increasing speed.

2. Suppose that the plane steady flow of an ideal fluid is bounded on one side by a plane wall MN and on the other by a free boundary AFB, on which there occurs a pressure discontinuity: p_1 on AF, p_2 on FB, with $p_1 < p_2$ (see Fig. 2). Construct the curve $A'F'B'$ which is the mirror image of AFB with respect to MN; the points of the strip $A'F'B'BFA$ correspond to the strip $|\psi| < Q$ in the plane w (here $Q = v_1 y_1 = v_2 y_2$ is the flux). The function $Z(w)$ satisfying the boundary conditions (taking $\varphi = 0$ at the point F)

$$\operatorname{Re} Z = 0 \quad \text{for} \quad \varphi > 0,\ \psi = \pm Q; \qquad \operatorname{Re} Z = 1 \quad \text{for} \quad \varphi < 0,\ \psi = \pm Q$$

is given by the Schwarz integral for the strip [2]:

$$Z(w) = \frac{1}{2Q} \int_{-\infty}^{0} \operatorname{sech} \frac{\pi(\varphi - w)}{2Q} \, d\varphi + iC = \frac{1}{\pi i} \ln \frac{e^{\pi w/2Q} + i}{e^{\pi w/2Q} - i} + iC \tag{2.1}$$

where that branch of the logarithm appears which satisfies $0 < \operatorname{Im} \ln(\ ) < \pi$ for $|\psi| < Q$, and the constant $C = 0$ if the axis of reals is directed along MN. From (1.1) and (2.1), setting $w(0) = 0$, one may express z by means of w and Z, to obtain

$$v_2 z = \int_0^w \left(\frac{e^{\pi w/2Q} + i}{e^{\pi w/2Q} - i}\right)^{ia} dw = -2Q \int_{1/2}^{Z} e^{-\pi a Z} \frac{dZ}{\sin \pi Z} \tag{2.2}$$

In order to determine z_0 (the position of the point F) it is necessary to evaluate the second integral occurring in Equation (2.2) between the limits $Z = 1/2$ and $Z = 1/2 - i\infty$, which, upon setting $Z = 1/2 - it/\pi$, becomes

$$z_0 = x_0 + iy_0 = \frac{2\sqrt{y_1 y_2}}{\pi}\left(-\int_0^\infty \frac{\sin at}{\cosh t}\,dt + i\int_0^\infty \frac{\cos at}{\cosh t}\,dt\right) \tag{2.3}$$

The first integral in (2.3) may be computed by means of a series, and the second can be evaluated in closed form [3]

$$x_0 = -\frac{4a\sqrt{y_1 y_2}}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2 + a^2} < 0, \qquad y_0 = \frac{2y_1 y_2}{y_1 + y_2} \tag{2.4}$$

Near the point F the formulas (2.1), (2.2), as was to be expected, reduce to the relations of Section 1 or $a > 0$. The speed decreases monotonically from v_1 to v_2, and takes the value $\sqrt{(v_1 v_2)}$ for $z = 0$, while the slope of the boundary with respect to MN increases monotonically from 0 to $+\infty$ on AF and decreases monotonically from $+\infty$ to 0 on FB (see Fig. 2).

By means of the second integral of (2.2), extended along the free boundary one may estimate the amount of twist (i.e. the distance d between boundary points at which the velocity of the fluid is parallel to the y axis). The ratio of this distance to the mean width of the channel of fluid for small a, is given by

$$\frac{d}{h} \approx \frac{32}{\pi^3}\, ae^{-\pi/2a} \qquad \left(h = \frac{y_1 + y_2}{2}\right) \tag{2.5}$$

and hence the distance decreases, as $a \to +0$, somewhat faster than the height $y_2 - y_1$ which is of the order πah.

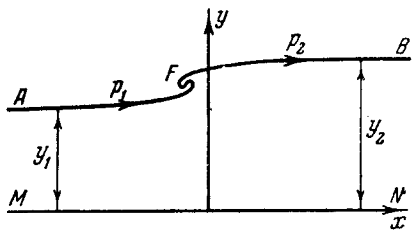

Fig. 2.

3. In a plane vortex free motion there is a velocity potential which satisfies Laplace's equation and is related to the pressure p by Cauchy's integral of the equations of motion:

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0 \tag{3.1}$$

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right] + \frac{p}{\rho} = 0$$

Consider an unsteady, self-similar motion and introduce the following change of variables

$$\xi = \frac{x}{ct^k},\ \eta = \frac{y}{ct^k};\ \varphi(x, y, t) = c^2 t^{2k-1}\Phi(\xi, \eta),\ p(x, y, t) = \rho c^2 t^{2k-2} P(\xi, \eta) \tag{3.2}$$

where c has the dimensions L/T^{-k}. In this last equation, $\Phi(\xi, \eta)$ is a harmonic function of its variables, satisfying on the free boundary the following two conditions (kinematic and dynamic)

$$\frac{\partial\Phi}{\partial\xi}\,d\eta - \frac{\partial\Phi}{\partial\eta}\,d\xi + k(\eta d\xi - \xi d\eta) = 0$$
$$(2k-1)\Phi - k\left(\xi\frac{\partial\Phi}{\partial\xi} + \eta\frac{\partial\Phi}{\partial\eta}\right) + \frac{1}{2}(\nabla\Phi)^2 + P_0 = 0 \qquad (3.3)$$

where P_0 is the nondimensional external pressure, which may be regarded as a discontinuous function on the boundary. Introducing the complex variables ζ and W ($\zeta = \xi + i\eta$, $\mathrm{Re}\,W = \Phi$), Equation (3.3) may be rewritten (with the asterisk denoting the complex conjugate number)

$$\mathrm{Im}\,(dW - k\zeta^* d\zeta) = 0, \quad \mathrm{Re}\left[(2k-1)W - k\zeta\frac{dW}{d\zeta}\right] + \frac{1}{2}\left|\frac{dW}{d\zeta}\right|^2 + P_0 = 0 \qquad (3.4)$$

In order to consider the motion in the neighborhood of the point of discontinuity ζ_0, where P_0 experiences a jump, let us put, in (3.4)

$$\zeta = \zeta_0 + \varepsilon z, \qquad W = W(\zeta_0) + \varepsilon k\zeta_0^* z + \varepsilon w \qquad (3.5)$$

where ε is a real parameter, and let us pass to the limit as $\varepsilon \to 0$, taking into account the boundedness of the derivative dw/dz (the velocity remains bounded in magnitude). The condition (3.4) becomes

$$\mathrm{Im}\,w = \mathrm{const}, \qquad \frac{1}{2}\left|\frac{dw}{dz}\right|^2 + P_0 = \mathrm{const}_1 \qquad (3.6)$$

which coincide with the conditions for the complex potential on the free boundary in the steady problem. Consequently, near the point of the pressure discontinuity on the boundary, the surface has a spiral form (Fig. 1) which deforms by similitude according to a power law in the time.

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